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C Pellegrini and F Petruccione

School of Physics and National Institute for Theoretical Physics, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa

E-mail: pelleg@math.univ-lyon1.fr and petruccione@ukzn.ac.za

Abstract

A non-Markovian model of quantum repeated interactions between a small quantum system and an infinite chain of quantum systems is presented. By adapting and applying usual projection operator techniques in this context, discrete versions of the integro-differential and time-convolutionless master equations for the reduced system are derived. Next, an intuitive and rigorous description of the indirect quantum measurement principle is developed and a discrete non-Markovian stochastic master equation for the open system is obtained. Finally, the question of unravelling in a particular model of non-Markovian quantum interactions is discussed.

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1. Introduction

The theory of open quantum systems describes the physical phenomena of dissipation and decoherence [12, 14, 16, 39, 40]. Starting from the microscopic formulation of the interaction between a small system and an environment in terms of the *Schrödinger equation*, there exist different ways to derive the *master equation* for the irreversible evolution of the small, i.e., reduced system. Typically, two approaches are considered: the Markovian and the non-Markovian one (see chapters 3, 9 and 10 in [14]). A very active line of research is focused on developing an appropriate description of indirect quantum measurements within both approaches. This research is motivated by recent experiments in quantum optics and quantum information [14, 26].

Physically, the Markovian approach is understood as a model without memory effects of the environment. In this setting, the master equation takes the form of a *Lindblad–Gorini–Kossakowski–Sudarshan equation* [27, 30], where the generator of the dynamics is a completely dissipative map [1, 14, 16, 27, 28, 30]. This equation is an ordinary differential equation, which describes the evolution of the state of the small system and the study of different physical phenomena: irreversibility, decoherence, return to equilibrium.

Mathematically, the approach makes use of the *Born–Markov approximation* or the *weak coupling limit*. Starting from the Hamiltonian description of the Schrödinger equation, tracing over the degrees of freedom of the environment and neglecting the memory of the interaction, the Markovian master equation is obtained.

In the Markovian context, measurement involves a stochastic perturbation of the Lindblad equation in terms of stochastic differential equations, i.e., *stochastic Schrödinger equations* [5, 7]. These equations have some remarkable properties. First, they conserve the purity of states, i.e. unravelling (the term 'stochastic Schrödinger equation' is mainly used when the equation preserved purity, otherwise it is called 'stochastic master equation'). Second, the expectation of the stochastic Schrödinger equation reproduces the dynamics induced by the Markovian master equation for the density matrix of the open system. These properties are very useful for the numerical simulation of the master equation. In fact, the so-called Monte Carlo wavefunction method is used extensively in quantum optics and quantum information [13, 15, 32].

In the non-Markovian approach, memory effects of the environment give rise to generalized master equations, i.e., integro-differential equations for the density operator of the open system. Usually, these equations are obtained by projection operator techniques, e.g. the *Nakajima–Zwanzig operator technique* [41] or the *time convolutioness operator technique* (see chapter 9 in [14]). These techniques allow for the formal description of more realistic models. However, the generalized master equations are difficult to manipulate [14, 37]. Even if formally exact analytical solutions can be obtained, these are very hard to solve numerically. In the non-Markovian context, the stochastic equations describing measurement procedures are usually expressed in terms of colored noise [19, 20] (the Markovian case involves only white noise). The justification of such models is far from being obvious and intuitive. Often, rigorous arguments are missing. Moreover, the question of non-Markovian unravelling is still highly debated [17–24, 38]. Especially, the interpretation of non-Markovian unravellings in terms of indirect measurement still remains an open problem.

In the Markovian case, a rigorous approach to the description of interaction and measurement lies in the theory of quantum stochastic calculus [33]. In this setup, the action of the environment (described by a Fock space) is modeled by quantum noises and the evolution is given by the solution of quantum stochastic differential equations [2, 25]. Quantum filtering theory [8, 9] is based on quantum stochastic calculus in order to describe quantum measurement and to derive stochastic Schrödinger equations. Recently, a discrete model called *quantum repeated interactions* has been introduced [3, 4]. This model provides a 'useful' approximation of the interaction between a small system and an environment. The model is a small system $\mathcal{H}_0$ in contact with an infinite chain of quantum systems representing the environment. All the elements of the chain are identical and independent. Pieces of the environment, denoted by $\mathcal{H}$, interact one after the other with $\mathcal{H}_0$ during a time τ. Hence, by renormalizing the interaction in terms of τ, quantum stochastic differential equation models can be obtained as continuous limits (τ goes to zero) of quantum repeated interactions models. This approach has been adapted to the context of measurement in [34–36]. It corresponds to the model of *quantum repeated measurements*. It has been shown that stochastic Schrödinger equations can be obtained as continuous limits of the discrete version of quantum measurement. Furthermore, via concrete procedures, the approach gives an intuitive and rigorous interpretation of quantum stochastic differential equations and stochastic Schrödinger equations.

The main aim of this paper is to present the non-Markovian model of quantum repeated interactions and discrete measurement. We define a clear mathematical model of the effect of the memory of the environment in this setup. Furthermore, we show that the natural projection

operator technique (Nakajima–Zwanzig, time convolutioness) used in the continuous non-Markovian approach can be adapted to the discrete context. We present a clear way to perform quantum repeated measurement in the non-Markovian case. As a result we obtain a rigorous discrete expression for the evolution of the small system with and without measurement. Finally, we investigate the problem of unravelling in this context. For a concrete model, we show that in the non-Markovian case unravelling imposes a Markovian structure except for some very special cases.

The paper is structured as follows. In section 2, we present the mathematical model of quantum repeated interactions in the non-Markovian case. We adapt the presentation of [3] to introduce memory effect of the infinite chain.

In section 3, we obtain the description of the evolution of the small system by computing the Nakajima–Zwanzig and time convolutioness projection operator technique. Especially, we obtain a discrete version of the evolution described in the previous investigations for the continuous case (see chapters 9 and 10 in [14] for all details concerning the continuous version).

Section 4 is devoted to the introduction of a model of measurement. We present a natural way to perform measurements in the context of non-Markovian quantum repeated interactions. We define a probabilistic setup describing the random evolution of the small system. By adapting the Nakajima–Zwanzig projection operator technique, we obtain a rigorous expression of an evolution equation which is a discrete version of the non-Markovian stochastic master equation. Next, we investigate the question of unravelling by studying a special case of non-Markovian quantum repeated interactions. We show that unravelling imposes strong assumptions for the evolution which in general lead to a Markovian dynamics.

2. Quantum repeated interaction model

This section is devoted to the description of the discrete model of quantum repeated interactions in the non-Markovian setup. We start by reviewing briefly the Markov treatment of quantum repeated interactions [3]. The canonical model is described as follows. A small system $\mathcal{H}_0$ is in contact with an infinite chain of identical and independent quantum systems (each element of the chain is denoted by $\mathcal{H}$). Each copy of $\mathcal{H}$ interacts with $\mathcal{H}_0$ in the following fashion. The first copy of $\mathcal{H}$ interacts with the small system during a time τ and disappears afterward. Then, the second copy interacts with $\mathcal{H}_0$ during the same time interval τ and so on. Physically, the fact that after each interaction the copy disappears is the Markov approximation.

Let us now describe the mathematical setup for the non-Markovian quantum repeated interactions. The main idea is to keep the memory of each interaction.

As the chain is supposed to be infinite, the state space of the chain is described as

$$T\Phi = \bigotimes_{j=1}^{\infty} \mathcal{H}_j,$$

where $\mathcal{H}_j \simeq \mathcal{H}$ for all k. To formulate the precise definition of this infinite tensor product we fix an orthonormal basis $\{X_0, \ldots, X_K\}$ of $\mathcal{H}$ (where $K + 1$ is the dimension of $\mathcal{H}$). The state $|X_0\rangle\langle X_0|$ can be regarded as the ground state. The basis of $T\Phi$ is constructed with respect to the stabilizing sequence induced by X_0.

To this end, let $\mathcal{P}$ be the set of subsets of the form $A = \{(n_1, i_1), \ldots, (n_k, i_k)\}$, where $k \in \mathbf{N}^\star$, $\{i_1, \ldots, i_k\} \in \{1, \ldots, K + 1\}^k$, and $n_1 < \cdots < n_k$ with $n_j \in \mathbf{N}^\star$. The basis of $T\Phi$ is denoted by $\overline{B} = \{X_A, A \in \mathcal{P}\}$, where for $A \in \mathcal{P}$ the vector X_A corresponds to

$$X_A = X_0 \otimes \cdots X_0 \otimes X_{i_1} \otimes X_0 \otimes \cdots X_0 \otimes X_{i_2} \otimes \cdots$$

and the vector X_{i_j} appears in the copy number n_j of $\mathcal{H}$.

Let us now define the basic operator. On $\mathcal{B}(\mathcal{H})$, the canonical operator with respect to $\{X_0, \ldots, X_k\}$ is denoted by a_{ij}, that is, for all $(i, j, k) \in \{0, K\}^3$ we have

$$a_{ij}(X_k) = \delta_{jk} X_i,$$

that is

$$a_{ij} = |X_i\rangle\langle X_j|,$$

in Dirac notations. By extension, we denote by $a_{ij}^{(k)}$ the operator acting as a_{ij} on $\mathcal{H}_k$, which is the copy number k of $\mathcal{H}$. On $T\Phi$, we have

$$a_{ij}^{(k)} = I \otimes \bigotimes_{j=1}^{k-1} I \otimes a_{ij}^k \otimes \bigotimes_{j>k} I.$$

The index k without round bracket means that the operator a_{ij} is set on the place number k is the infinite tensor product.

Hence, the coupled system, system and chain, is described by the Hilbert space

$$\Gamma = \mathcal{H}_0 \otimes T\Phi. \tag{1}$$

We endow this Hilbert space with the following state:

$$\mu = \rho \otimes \bigotimes_{j=1}^{\infty} \beta_j, \tag{2}$$

where ρ is a reference state of $\mathcal{H}_0$ and $\beta_j = \beta$ for all j, with $\beta = |X_0\rangle\langle X_0| = a_{00}$ is the reference state of $\mathcal{H}$ (this corresponds to a system at zero temperature).

Let us now describe the interaction setup. The first copy of $\mathcal{H}$ interacts with $\mathcal{H}_0$ during a time τ. After this interaction a second copy of $\mathcal{H}$ interacts with $\mathcal{H}_0$ and the first copy, that is $\mathcal{H}_2$ interacts with $\mathcal{H}_0 \otimes \mathcal{H}_1$ and so on. As a consequence, the kth copy of $\mathcal{H}$ interacts with $\mathcal{H}_0 \otimes \bigotimes_{j=1}^{k-1} \mathcal{H}_j$, that is we keep the memory of the previous interactions.

Mathematically, we consider a sequence of unitary operators (U_k) for $k \geqslant 1$. For each k the operator U_k acts non-trivially on $\mathcal{H}_0 \otimes \bigotimes_{j=1}^{k} \mathcal{H}_j$ and acts like the identity operator on $\bigotimes_{j\geqslant k+1} \mathcal{H}_j$. The sequence of unitary operators (V_k) which describes the repeated quantum interactions is defined by putting

$$V_k = U_k \cdots U_1 \tag{3}$$

for all k (the operator V_k describes the k first interactions). Hence, in the Schrödinger picture, after k interactions, the initial state μ defined by (2) becomes

$$\mu_k = V_k \mu V_k^\star. \tag{4}$$

It is straightforward to see that $\mu_{k+1} = U_{k+1} \mu_k U_{k+1}^\star$. The sequence (μ_k) describes the evolution of the system in the quantum repeated interaction setup with memory.

It is important to note that in the Markovian case the unitary operator U_k acts only non-trivially on the tensor product of $\mathcal{H}_0$ with $\mathcal{H}_k$, which is the kth copy of $\mathcal{H}$. On the rest of the Hilbert space, it acts like the identity operator. In the homogeneous case for example, the operator U_k can be expressed, for all k, as

$$U_k = \sum_{i,j=0}^{K} U_{ij} \otimes a_{ij}^{(k)}, \tag{5}$$

where U_{ij} are operators on $\mathcal{H}_0$. In the following, to ease the notation, we suppress the symbol $\otimes$ in similar expression.

In order to generalize the above class of Markovian models, which have been studied extensively in [3], we apply the classical projection operator technique of Nakajima–Zwanzig and its time-convolutionless version to the non-Markovian quantum repeated interactions model (see chapter 9 in [14] for a general introduction for Nakajima–Zwanzig and time-convolutionless operator technique).

3. Discrete non-Markov evolution equation

This section is devoted to the description of the discrete evolution of the small system in the non-Markovian case. In section 3.1, we apply the Nakajima–Zwanzig operator technique and in section 3.2, we investigate the equivalent time-convolutionless projection operator technique for the discrete case.

3.1. The Nakajima–Zwanzig projection operator technique

We start by applying the Nakajima–Zwanzig projection operator technique to the sequence (μ_k) introduced in (4). For any state α on Γ, we define the Nakajima–Zwanzig operators:

$$\mathcal{P}\alpha = \mathrm{Tr}_{T\Phi}[\alpha] \otimes \bigotimes_{j=1}^{\infty} \beta_j$$
$$\mathcal{Q}\alpha = \alpha - \mathcal{P}\alpha, \tag{6}$$

where $\mathrm{Tr}_{T\Phi}[\alpha]$ represents the partial trace of α with respect to the chain. In the canonical approach, the operator $\mathcal{P}$ projects onto the relevant part of the small system. The aim is to obtain an evolution equation which describes the sequence $(\mathcal{P}\mu_k)$ representing the evolution of the relevant part of $\mathcal{H}_0$.

For all operators α on $\mathcal{B}(\Gamma)$, let us denote

$$\mathcal{L}_k(\alpha) = U_k \alpha U_k^{\star}. \tag{7}$$

This way, for a fixed k, the projection operators $\mathcal{P}\mu_{k+1}$ and $\mathcal{Q}\mu_{k+1}$ are given by

$$\mathcal{P}\mu_{k+1} = \mathcal{P}U_{k+1}\mu_k U_{k+1}^{\star} = \mathcal{P}U_{k+1}(\mathcal{P}+\mathcal{Q})\mu_k U_{k+1}^{\star}$$
$$= \mathcal{P}\mathcal{L}_{k+1}(\mathcal{P}\mu_k) + \mathcal{P}\mathcal{L}_{k+1}(\mathcal{Q}\mu_k), \tag{8}$$

$$\mathcal{Q}\mu_{k+1} = \mathcal{Q}\mathcal{L}_{k+1}(\mathcal{P}\mu_k) + \mathcal{Q}\mathcal{L}_{k+1}(\mathcal{Q}\mu_k). \tag{9}$$

Iterating (9) and taking into account the non-commutativity of the operator $\mathcal{L}_k$, we obtain the following expression for $\mathcal{Q}\mu_{k+1}$:

$$\mathcal{Q}\mu_{k+1} = \sum_{i=0}^{k}\left(\mathcal{Q}\mathcal{L}_{k+1}\left(\mathrm{T}_{\leftarrow}\left(\prod_{j=i}^{k}\mathcal{Q}\mathcal{L}_j\right)(\mathcal{P}(\mu_i))\right)\right) + \mathcal{Q}\mathcal{L}_{k+1}\left(\mathrm{T}_{\leftarrow}\left(\prod_{j=1}^{k}\mathcal{Q}\mathcal{L}_j\right)(\mathcal{Q}(\mu_0))\right). \tag{10}$$

As usual, the operator $\mathrm{T}_{\leftarrow}$ describes the chronological time ordering (the discrete time arguments j increase from right to left). This expression corresponds to a discrete version of the time ordering exponential term appearing in the non-Markovian continuous case (see [14]). Obviously, the first term $\mathcal{Q}(\mu_0) = 0$. Hence, by replacing expression (10) in expression (8), we get

$$\mathcal{P}\mu_{k+1} = \mathcal{P}\mathcal{L}_{k+1}(\mathcal{P}\mu_k) + \mathcal{P}\mathcal{L}_{k+1}\left(\sum_{i=0}^{k-1}\mathcal{Q}\mathcal{L}_k\left(\mathrm{T}_{\leftarrow}\left(\prod_{j=i}^{k-1}\mathcal{Q}\mathcal{L}_j\right)(\mathcal{P}\mu_i)\right)\right)$$

$$= \mathcal{P}\mathcal{L}_{k+1}(\mathcal{P}\mu_k) + \sum_{i=0}^{k-1}\mathcal{P}\mathcal{L}_{k+1}\left(\mathrm{T}_{\leftarrow}\left(\prod_{j=i}^{k}\mathcal{Q}\mathcal{L}_j\right)(\mathcal{P}\mu_i)\right). \tag{11}$$

The above expression describes the complete time evolution of the relevant part of $\mathcal{H}_0$. Similarly in the continuous case, it appears as a discrete integro-differential equation which involves all the history of the evolution.

It might be interesting to see how the Markovian description emerges from the more general approach described above. As in the continuous case, we obtain a discrete version of the Lindblad, Gorini, Kossakowski, Sudarshan evolution equation as is shown in the following proposition.

Proposition 1. *Let (U_k) be a sequence of unitary operators which describe a Markovian evolution. For all k, for all $i < k$ and for all states γ on Γ, we have*

$$\mathcal{P}\mathcal{L}_{k+1}\left(\mathrm{T}_{\leftarrow}\left(\prod_{j=i}^{k}\mathcal{Q}\mathcal{L}_j\right)(\mathcal{P}\gamma)\right) = 0. \tag{12}$$

Hence expression ([11](#)) becomes

$$\mathcal{P}\mu_{k+1} = \mathcal{P}\mathcal{L}_{k+1}(\mathcal{P}\mu_k).$$

Furthermore, in the Markovian homogeneous case ([5](#)) there exists a completely positive map $\mathcal{L}$ acting on $\mathcal{B}(\mathcal{H}_0)$, such that for all k

$$\mathcal{P}\mu_{k+1} = \mathcal{L}(\mathcal{P}\mu_k). \tag{13}$$

Proof. Let us start by showing the last part of the proposition. Recall that the operator (U_k), in the homogeneous Markovian case ([5](#)), can be expressed as

$$U_k = \sum_{ij=0}^{K} U_{ij} a_{ij}^{(k)}.$$

Since, for all k, the operator V_k, defined in equation ([3](#)) acts only on the k first copies of $\mathcal{H}$, it is worth noticing that for all k

$$\mu_k = \mathrm{Tr}_{\bigotimes_{j>k}\mathcal{H}_j}[\mu_k] \otimes \bigotimes_{j=k+1}^{\infty} a_{00}^j.$$

Hence for all k and all $X \in \mathcal{B}(\mathcal{H}_0)$

$$\mathrm{Tr}[\mathrm{Tr}_{T\Phi}[\mu_{k+1}]X] \tag{14}$$

$$= \mathrm{Tr}\left[\mu_{k+1}X \otimes \bigotimes_{j=1}^{\infty} I\right] = \mathrm{Tr}\left[\mu_k U_{k+1}^{\star}\left(X \otimes \bigotimes_{j=1}^{\infty} I\right)U_{k+1}\right]$$

$$= \mathrm{Tr}\left[\mu_k \sum_{i,j,p} U_{ij}^{\star}XU_{pj}a_{ip}^{(k+1)}\right]$$

$$= \mathrm{Tr}\left[\mathrm{Tr}_{\otimes_{j>k} \mathcal{H}_j}[\mu_k] \otimes \bigotimes_{j=k+1}^{\infty} a_{00}^j \left(\sum_{i,j,p} U_{ij}^\star X U_{pj} a_{ip}^{(k+1)} \right) \right]$$

$$= \mathrm{Tr}\left[\mathrm{Tr}_{\otimes_{m>k} \mathcal{H}_m}[\mu_k] \otimes \bigotimes_{m>k+1} a_{00}^m \left(\sum_{j,p} U_{0j}^\star X U_{pj} \otimes \bigotimes_{j=1}^{k} I \otimes a_{0p}^{k+1} \otimes \bigotimes_{j>k+1} a_{00}^j \right) \right]$$

$$= \sum_{j,p} \left(\mathrm{Tr}\left[\mathrm{Tr}_{\otimes_{m>k} \mathcal{H}_m}[\mu_k] \left(U_{0j}^\star X U_{pj} \otimes \bigotimes_{j=1}^{k} I \right) \right] \mathrm{Tr}\left[a_{0p}^{k+1} \otimes \bigotimes_{j>k+1} a_{00}^j \right] \right)$$

$$= \sum_{j} \left(\mathrm{Tr}\left[\mathrm{Tr}_{\otimes_{m>k} \mathcal{H}_m}[\mu_k] \left(U_{0j}^\star X U_{0j} \otimes \bigotimes_{j=1}^{k} I \right) \right] \right)$$

$$= \mathrm{Tr}\left[\mathrm{Tr}_{T\Phi}[\mu_k] \left(\sum_{j} U_{0j}^\star X U_{0j} \right) \right]$$

$$= \mathrm{Tr}\left[\left(\sum_{j} U_{0j} (\mathrm{Tr}_{T\Phi}[\mu_k]) U_{0j}^\star \right) X \right] \tag{15}$$

$$= \mathrm{Tr}[\mathcal{L}(\mathrm{Tr}_{T\Phi}[\mu_k]) X]. \tag{16}$$

This proves that $\mathcal{P}\mu_{k+1} = \mathcal{L}(\mathcal{P}\mu_k)$. It is interesting to note that we have obtained the Kraus decomposition (15) of the completely positive map $\mathcal{L}$ (16) (see [2, 14] for more details). This result was obtained in a different way in [3]. The equivalent proposition for the non-homogeneous case can simply be obtained by replacing the terms U_{ij} by the non-homogeneous terms $U_{ij}^{(k+1)}$.

Now we turn our attention to the first part of the proposition. Equation (12) follows from the fact that for all operators α of the form

$$\alpha = \eta \otimes \bigotimes_{j=k+1}^{\infty} a_{00}^j, \tag{17}$$

with η being any operator on $\mathcal{B}\left(\mathcal{H}_0 \otimes \bigotimes_{j=1}^{k} \mathcal{H}_j\right)$, we have

$$\mathcal{P} U_{k+1} \mathcal{Q} \alpha U_{k+1}^\star = 0. \tag{18}$$

Indeed, with the definition of the operation $\mathcal{P}$ and the unitary operators (U_k), it is straightforward to see that the operator $\mathrm{T}_{\leftarrow}\left(\prod_{j=i}^{k} \mathcal{Q}\mathcal{L}_j \right)(\mathcal{P}\gamma)$ is of the same form as an operator α (17).

Now, we are in the position to prove the result (18). The operator α can be expressed as

$$\alpha = \sum_{(i_1,j_1),...,(i_k,j_k)\in\{0,...,K\}} \alpha_{(i_1,j_1),...,(i_k,j_k)} \otimes a_{i_1,j_1}^1 \otimes \cdots \otimes a_{i_k,j_k}^k \otimes \bigotimes_{j>k} a_{00}^j.$$

Then, with our notation and with the rule of the partial trace, one can see that

$$\mathcal{P}\alpha = \sum_{i_1,...,i_k} \alpha_{(i_1,i_1),...,(i_k,i_k)} \otimes \bigotimes_{j=1}^{\infty} a_{00}^j.$$

Accordingly, for $\mathcal{Q}\alpha$, we obtain

$$\mathcal{Q}\alpha = \sum_{(i_1,j_1),...,(i_k,j_k)\in\{0,...,K\}} \beta_{(i_1,j_1),...,(i_k,j_k)} \otimes a_{i_1,j_1}^1 \otimes \cdots \otimes a_{i_k,j_k}^k \otimes \bigotimes_{j>k} a_{00}^j, \tag{19}$$

with

$$\beta_{(0,0),\dots,(0,0)} = - \sum_{(i_1,\dots,i_k)\neq(0,\dots,0)} \alpha_{(i_1,i_1),\dots,(i_k,i_k)},$$

$$\beta_{(i_1,j_1),\dots,(i_k,j_k)} = \alpha_{(i_1,j_1),\dots,(i_k,j_k)}, \qquad \text{for all} \quad (i_1,j_1),\dots(i_k,j_k) \neq (0,0),\dots,(0,0).$$

By applying $\mathcal{L}_{k+1}$ on $\mathcal{Q}\alpha$ (19), we get

$$U_{k+1}\mathcal{Q}\alpha U_{k+1}^\star$$
$$= \sum_{ij} \sum_{(i_1,j_1),\dots,(i_k,j_k)\in\{0,\dots,K\}} U_{i0}\beta_{(i_1,j_1),\dots,(i_k,j_k)}U_{j0}^\star \otimes a_{i_1,j_1}^1 \otimes \dots \otimes a_{i_k,j_k}^k \otimes a_{ij} \otimes \bigotimes_{j>k+1} a_{00}^j.$$

The result (18) is proved by noting that

$$\mathcal{P}U_{k+1}\mathcal{Q}\alpha U_{k+1}^\star = \sum_i \sum_{i_1,\dots,i_k\in\{0,\dots,K\}} U_{i0}\beta_{(i_1,i_1),\dots,(i_k,i_k)}U_{i0}^\star \otimes \bigotimes_{j=1}^\infty a_{00}^j$$

$$= \sum_i U_{i0}\beta_{(0,0)\dots(0,0)}U_{i0}^\star \otimes \bigotimes_{j=1}^\infty a_{00}^j$$

$$+ \sum_i U_{i0} \sum_{i_1,\dots,i_k\neq0,\dots,0} \alpha_{(i_1,i_1),\dots,(i_k,i_k)}U_{i0}^\star \otimes \bigotimes_{j=1}^\infty a_{00}^j$$

$$= 0.$$

This completes the proof of the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

3.2. The time-convolutioness projection operator method

In this section, we apply the time-convolutionless version of the Nakajima–Zwanzig operator technique to the framework of non-Markovian quantum repeated interactions. The aim is to derive a discrete time-evolution equation for $(\mathcal{P}\mu_k)$ which links $\mathcal{P}\mu_{k+1}$ only to $\mathcal{P}\mu_k$.

To this end, for all p, we define the inverse of the operator $\mathcal{L}_p$ as

$$\mathcal{L}_p^{(-1)}(\alpha) = (U_p^{-1})\alpha(U_p^{-1})^\star,$$

for all operators α. It is straightforward to see that for all p we have $\mu_p = \mathcal{L}_{p+1}^{(-1)}(\mu_{p+1})$. Hence, expression (10) obtained in the previous section becomes

$$\mathcal{Q}\mu_{k+1} = \sum_{i=0}^k \mathcal{Q}\mathcal{L}_{k+1}\left(\mathrm{T}_\leftarrow\left(\prod_{j=i}^k \mathcal{Q}\mathcal{L}_j\right)\mathcal{P}\mathrm{T}_\rightarrow\left(\prod_{p=i+1}^{k+1}\mathcal{L}_p^{(-1)}\right)(\mu_{k+1})\right)$$

$$= \sum_{i=0}^k \mathcal{Q}\mathcal{L}_{k+1}\left(\mathrm{T}_\leftarrow\left(\prod_{j=i}^k \mathcal{Q}\mathcal{L}_j\right)\mathcal{P}\mathrm{T}_\rightarrow\left(\prod_{p=i+1}^{k+1}\mathcal{L}_p^{(-1)}\right)(\mathcal{P}+\mathcal{Q})(\mu_{k+1})\right)$$

$$= \sum_{i=0}^k \mathcal{Q}\mathcal{L}_{k+1}\left(\mathrm{T}_\leftarrow\left(\prod_{j=i}^k \mathcal{Q}\mathcal{L}_j\right)\mathcal{P}\mathrm{T}_\rightarrow\left(\prod_{p=i+1}^{k+1}\mathcal{L}_p^{(-1)}\right)(\mathcal{P}\mu_{k+1})\right)$$

$$+ \sum_{i=0}^k \mathcal{Q}\mathcal{L}_{k+1}\left(\mathrm{T}_\leftarrow\left(\prod_{j=i}^k \mathcal{Q}\mathcal{L}_j\right)\mathcal{P}\mathrm{T}_\rightarrow\left(\prod_{p=i+1}^{k+1}\mathcal{L}_p^{(-1)}\right)(\mathcal{Q}\mu_{k+1})\right)$$

$$= \sum_{i=0}^k \mathrm{T}_\leftarrow\left(\prod_{j=i-1}^{k+1}\mathcal{Q}\mathcal{L}_j\right)\mathcal{Q}\mathcal{L}_i\mathcal{P}\mathrm{T}_\rightarrow\left(\prod_{p=i+1}^{k+1}\mathcal{L}_p^{(-1)}\right)(\mathcal{P}\mu_{k+1})$$

$$+ \sum_{i=0}^{k} \mathrm{T}_{\leftarrow} \left(\prod_{j=k+1}^{i-1} \mathcal{Q}\mathcal{L}_j \right) \mathcal{Q}\mathcal{L}_i \mathcal{P}\mathrm{T}_{\rightarrow} \left(\prod_{p=i+1}^{k+1} \mathcal{L}_p^{(-1)} \right) (\mathcal{Q}\mu_{k+1}), \tag{20}$$

where $\mathrm{T}_{\rightarrow}$ corresponds to the time antichronological ordering. For a fixed m, we define for all operators $\alpha \in \mathcal{B}(\Gamma)$ the following super-operator:

$$\mathcal{K}_{m+1}(\alpha) = \sum_{i=0}^{m} \mathrm{T}_{\leftarrow} \left(\prod_{j=m+1}^{i-1} \mathcal{Q}\mathcal{L}_j \right) \mathcal{Q}\mathcal{L}_i \mathcal{P}\mathrm{T}_{\rightarrow} \left(\prod_{p=i+1}^{m+1} \mathcal{L}_p^{(-1)} \right) (\alpha).$$

Expression (20) becomes

$$[I - \mathcal{K}_{k+1}]\mathcal{Q}\mu_{k+1} = \mathcal{K}_{k+1}\mathcal{P}\mu_{k+1}. \tag{21}$$

Hence, if we assume that $[I - \mathcal{K}_{k+1}]$ is invertible, we get

$$\mathcal{Q}(\mu_{k+1}) = [I - \mathcal{K}_{k+1}]^{(-1)}\mathcal{K}_{k+1}\mathcal{P}\mu_{k+1}. \tag{22}$$

As a consequence, for the expression of $\mathcal{P}\mu_{k+1}$ can now be written as

$$\mathcal{P}\mu_{k+1} = \mathcal{P}\mathcal{L}_{k+1}\mathcal{P}\mu_k + \mathcal{P}\mathcal{L}_{k+1}\big[[I - \mathcal{K}_k]^{(-1)}\mathcal{K}_k\mathcal{P}\mu_k\big]. \tag{23}$$

The expression (23) is the discrete time-convolutionless equation of evolution for the reduced system. The invertibility of $I - \mathcal{K}_k$ implies the discrete equivalent of the locality in the time continuous case (see chapter 9 in [14] for more details).

4. Non Markov quantum repeated interactions with quantum measurement

In the Markovian case, the theory of quantum measurement and the corresponding evolution are well understood and well studied. This is not the case in a non-Markovian set up. One of the open questions concerns the description of the evolution of the reduced system undergoing indirect quantum measurement. The way to describe the random non-Markovian evolution of the reduced system in the continuous time case is far from obvious and in general it is based on very technical aspect [20, 23].

In this section, we present a very clear mathematical and physical way to describe indirect quantum measurement in the discrete non-Markovian setup. The approach consists of adapting the quantum repeated measurement [8, 34–36] in the non-Markovian quantum repeated interactions model. In the same spirit of these papers, we will show how to derive a discrete non-Markovian stochastic master equation. Let us stress that in [34–36] the Markovian stochastic master equations are obtained as a time continuous limit of such discrete models.

We proceed in the following way. After each interaction, implying a new copy of $\mathcal{H}$, a quantum measurement is performed on the last copy which has interacted (this corresponds to the natural indirect measurement scheme introduced in [8, 34–36]). Each measurement involves a random evolution of the state of the system Γ. The repeated sequence of measurement gives rise to a random sequence of states on Γ. Then, we apply the Nakajima–Zwanzig operator technique on this sequence in order to obtain the evolution of the relevant part of the reduced system $\mathcal{H}_0$. In fact, this strategy is very natural and does not require any phenomenological inputs.

After this, we will consider the problem of unravelling. Obtaining unravelling in the non-Markovian case imposes strong assumptions on the interaction setup. Focusing on a special case, we show that essentially unravelling implies the Markov assumption except for very special situations.

4.1. Quantum repeated measurements

We start by describing the setup of quantum repeated measurements on the whole system. After each interaction, a measurement of an observable is performed on the last copy of $\mathcal{H}$ which has interacted.

We need to introduce some notations. Let $A = \sum_{i=0}^{p} \lambda_i P_i$ be the spectral decomposition of an observable of $\mathcal{H}$. In analogy to the construction of the basic operators $a_{ij}^{(k)}$ in section 2, we introduce

$$A^{(k)} = I \otimes \bigotimes_{j=1}^{k-1} I \otimes A \otimes \bigotimes_{j>k+1} I.$$

The operator $A^{(k)}$, which is an observable of $\mathcal{H}_k$, is the extension of the observable A to the whole space Γ (1). In the same way, for all $i \in \{0, \ldots, p\}$ we denote the eigen-projectors as

$$P_i^{(k)} = I \otimes \bigotimes_{j=1}^{k-1} I \otimes P_i \otimes \bigotimes_{j>k+1} I.$$

Let us describe a single interaction and a single measurement. After the first interaction the state of the system is described by $\mu_1 = U_1 \mu U_1^\star$, where μ is the initial state (2). According to the postulates of quantum measurement only the eigenvalues of $A^{(1)}$ can be observed; the result is random and obeys to the probability law

$$P[\text{to observe} \lambda_i] = \text{Tr}\big[\mu_1 P_i^{(1)}\big].$$

Furthermore, if we have observed the eigenvalue λ_i, the state of the system becomes

$$\rho_1(i) = \frac{P_i^{(1)} \mu_1 P_i^{(1)}}{\text{Tr}\big[\mu_1 P_i^{(1)}\big]}.$$

This is the so-called *von Neumann reduction postulate* (more general measurement procedures are described by instruments [6, 7]). The new state ρ_1 is actually a random variable. It describes the result of one interaction and one measurement.

Let us make precise the probabilistic framework of the complete procedure of quantum repeated measurements. To this end, we introduce the probabilistic space

$$\Sigma = \{0, 1, \ldots, p\}^{\mathbf{N}^\star},$$

where the index i corresponds to the eigenvalue λ_i. We endow this space with the cylinder algebra $\mathcal{C}$ generated by the set

$$\Lambda_{i_1,\ldots,i_k} = \{\omega \in \Sigma \,|\, \omega_1 = i_1, \ldots, \omega_k = i_k\}.$$

Now, we define a probability law on the cylinder set. To this end, we introduce for all $(i_1, \ldots, i_k) \in \{0, \ldots, p\}^k$:

$$\tilde{\mu}_k(i_1, \ldots, i_k) = \text{T}_\leftarrow \left(\prod_{j=1}^{k} P_{i_j}^{(j)} U_j \right) (\mu) \text{T}_\rightarrow \left(\prod_{j=1}^{k} U_j^\star P_{i_j}^{(j)} \right). \tag{24}$$

To simplify the above expression, we introduce the notation

$$\overline{P}_{i_j}^{(j)}[\alpha] = P_{i_j}^{(j)} \alpha P_{i_j}^{(j)}.$$

Hence, by using the definition of the operations $\mathcal{L}_k$, we get

$$\tilde{\mu}_k(i_1, \ldots, i_k) = \text{T}_\leftarrow \left(\prod_{j=1}^{k} \overline{P}_{i_j}^{(j)} \mathcal{L}_j \right) (\mu). \tag{25}$$

This corresponds to the non-normalized state, if we have observed the eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_k}$. The probability law on the cylinder set is now defined as

$$P[\Lambda_{i_1,\ldots,i_k}] = \mathrm{Tr}[\tilde{\mu}_k(i_1,\ldots,i_k)].$$

It is easy to check that the above definition satisfies the Kolmogorov consistency criterion. As a consequence, it defines a unique probability law P on $(\Sigma, \mathcal{C})$. Now for all k and all $\omega \in \Sigma$ we can define the following random variable:

$$\rho_k(\omega) = \frac{\tilde{\mu}_k(\omega_1, \ldots, \omega_k)}{\mathrm{Tr}[\tilde{\mu}_k(\omega_1, \ldots, \omega_k)]}.$$

The quantum repeated interactions combined with quantum repeated measurements are then described by the random sequence (ρ_k). Such a sequence is called a *discrete quantum trajectory* (on the whole space). To complete the description of discrete quantum trajectories on the whole space, we have the following Markov property.

Proposition 2. *The random sequence (ρ_k) is a Markov chain on $(\Omega, \mathcal{C}, P)$ valued in the set of states on Γ.*

More precisely, let θ be a state on Γ, if $\rho_k = \theta$ the random variable ρ_{k+1} takes the values

$$\rho_{k+1} = \frac{\overline{P}_i^{(k+1)}\mathcal{L}_{k+1}(\theta)}{\mathrm{Tr}[\overline{P}_i^{(k+1)}\mathcal{L}_{k+1}(\theta)]}, \qquad i = 0, \ldots, p,$$

with probability $p_i^{k+1} = \mathrm{Tr}[\overline{P}_i^{(k+1)}\mathcal{L}_{k+1}(\theta)]$.

In the following section, we apply the Nakajima–Zwanzig projection operator technique to the random sequence (ρ_k). As in the case without measurement (see section 3.1), this allows for the description of the evolution of the relevant part of the system. The projection on the reduced system gives rise to a random sequence of states of $\mathcal{H}_0$, which is, in general, non-Markovian.

4.2. Non-Markovian stochastic discrete evolution equation

This section is devoted to the description of the discrete stochastic evolution equation of the reduced system obtained by the Nakajima–Zwanzig projection operator technique. The equation we obtain is a discrete version of the non-Markovian stochastic master equations (see [20, 23] for continuous version).

Let (ρ_k) be a quantum trajectory as described in proposition 3. For all k and for all $i \in \{0, \ldots, p\}$ we denote the transition probabilities by $p_i^{k+1} = \mathrm{Tr}[\overline{P}_i^{(k+1)}\mathcal{L}_{k+1}(\rho_k)]$. Hence, with proposition 2, we can describe the evolution of (ρ_k) by the following equation:

$$\rho_{k+1}(\omega) = \sum_{i=0}^{p} \frac{(\overline{P}_i^{k+1}\mathcal{L}_{k+1})(\rho_k(\omega))}{p_i^{k+1}} \mathbf{1}_i^{k+1}(\omega), \tag{26}$$

for all $\omega \in \Omega$. In (26), the indicator function $\mathbf{1}_i^{k+1}$ corresponds to the observation of the eigenvalue λ_i during the $(k+1)$ th measurement. More precisely, we have $\mathbf{1}_i^{k+1}(\omega) = \mathbf{1}_i(\omega_{k+1})$, for all $\omega \in \Omega$ (in the following, we suppress the argument ω to enlighten the notations). Let us stress that we suppose implicitly that the quotient $\mathbf{1}_i^{k+1}/p_i^{k+1}$ is equal to zero if $p_i^{k+1} = 0$; this is consistent with respect to the probability P on Ω. By applying the Nakajima–Zwanzig operator $\mathcal{P}$ and $\mathcal{Q}$ introduced in section 3.1 to (26), the relation $I = \mathcal{P} + \mathcal{Q}$ gives us

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$$\mathcal{P}\rho_{k+1} = \mathcal{P}\left(\sum_{i=0}^{p} \frac{(\overline{P}_i^{k+1}\mathcal{L}_{k+1})(\rho_k)}{p_i^{k+1}}\mathbf{1}_i^{k+1}\right)$$

$$= \sum_{i=0}^{p} \frac{1}{p_i^{k+1}}\mathcal{P}(\overline{P}_i^{k+1}\mathcal{L}_{k+1})(\mathcal{P}\rho_k)\mathbf{1}_i^{k+1} + \sum_{i=0}^{p} \frac{1}{p_i^{k+1}}\mathcal{P}(\overline{P}_i^{k+1}\mathcal{L}_{k+1})(\mathcal{Q}\rho_k)\mathbf{1}_i^{k+1},$$

$$\mathcal{Q}\rho_{k+1} = \mathcal{Q}\left(\sum_{i=0}^{p} \frac{(\overline{P}_i^{k+1}\mathcal{L}_{k+1})(\rho_k)}{p_i^{k+1}}\mathbf{1}_i^{k+1}\right)$$

$$= \sum_{i=0}^{p} \frac{1}{p_i^{k+1}}\mathcal{Q}(\overline{P}_i^{k+1}\mathcal{L}_{k+1})(\mathcal{P}\rho_k)\mathbf{1}_i^{k+1} + \sum_{i=0}^{p} \frac{1}{p_i^{k+1}}\mathcal{Q}(\overline{P}_i^{k+1}\mathcal{L}_{k+1})(\mathcal{Q}\rho_k)\mathbf{1}_i^{k+1}. \tag{27}$$

By iteration, as is done in the case without measurement, we get the following expression for $\mathcal{Q}\rho_{k+1}$:

$$\mathcal{Q}\rho_{k+1} = \sum_{j=1}^{k+1} \sum_{(i_{k+1},\dots,i_j)\in\{0,\dots,p\}} \mathrm{T}_{\leftarrow}\left(\prod_{l=j}^{k+1}\mathcal{Q}(\overline{P}_{i_l}^{l}\mathcal{L}_l)\right)(\mathcal{P}\rho_{j-1})\prod_{l=k+1}^{j}\frac{\mathbf{1}_{i_l}^{l}}{p_{i_l}^{l}}$$

$$+ \sum_{(i_{k+1},\dots,i_1)\in\{0,\dots,p\}} \mathrm{T}_{\leftarrow}\left(\prod_{l=1}^{k+1}\mathcal{Q}(\overline{P}_{i_l}^{l}\mathcal{L}_l)\right)(\mathcal{Q}\rho_0)\prod_{l=k+1}^{1}\frac{\mathbf{1}_{i_l}^{l}}{p_{i_l}^{l}}$$

$$= \sum_{j=1}^{k+1} \sum_{(i_{k+1},\dots,i_j)\in\{0,\dots,p\}} \mathrm{T}_{\leftarrow}\left(\prod_{l=j}^{k+1}\mathcal{Q}(\overline{P}_{i_l}^{l}\mathcal{L}_l)\right)(\mathcal{P}\rho_{j-1})\prod_{l=k+1}^{j}\frac{\mathbf{1}_{i_l}^{l}}{p_{i_l}^{l}}. \tag{28}$$

The last equality in (28) comes from the fact that $\mathcal{Q}(\rho_0) = \mathcal{Q}(\mu_0) = 0$. Hence, we get the following equation:

$$\mathcal{P}\rho_{k+1} = \sum_{i=0}^{p} \frac{1}{p_i^{k+1}}\mathcal{P}(\overline{P}_i^{k+1}\mathcal{L}_{k+1})(\mathcal{P}\rho_k)\mathbf{1}_i^{k+1} + \sum_{j=1}^{k} \sum_{(i_{k+1},i_k,\dots,i_1)\in\{0,\dots,p\}} \mathcal{P}(\overline{P}_{i_{k+1}}^{k+1}\mathcal{L}_{k+1})$$

$$\times \left(\mathrm{T}_{\leftarrow}\left(\prod_{l=j}^{k}\mathcal{Q}(\overline{P}_{i_l}^{l}\mathcal{L}_l)(\mathcal{P}\rho_{j-1})\right)\right)\prod_{l=k}^{j}\frac{\mathbf{1}_{i_l}^{l}}{p_{i_l}^{l}}. \tag{29}$$

This expression is valid for $k \geqslant 1$. For $k = 0$, the expression is given by equation (27). In order to express the above equation only with terms $\mathcal{P}\mu_i$, it is worth noting that for all k and for all i

$$p_i^{k+1} = \mathrm{Tr}\big[\overline{P}_i^{(k+1)}\mathcal{L}_{k+1}(\rho_k)\big]$$

$$= \mathrm{Tr}\big[\mathcal{P}(\overline{P}_i^{(k+1)}\mathcal{L}_{k+1}(\rho_k))\big]$$

$$= \mathrm{Tr}\big[\mathcal{P}(\overline{P}_i^{(k+1)}\mathcal{L}_{k+1}(\mathcal{P}\rho_k))\big] + \mathrm{Tr}\big[\mathcal{P}(\overline{P}_i^{(k+1)}\mathcal{L}_{k+1}(\mathcal{Q}\rho_k))\big].$$

By replacing the expression of $\mathcal{Q}\rho_k$ with (28), we get an expression involving only terms $\mathcal{P}\mu_i$.

Expression (29) describes then the random evolution of the relevant part of the system. It is typically a non-Markovian chain evolution, because the expression at time k involves all the past of the sequence. Equation (29) is the discrete version of the non-Markovian stochastic master equation. The sequence $(\mathcal{P}\mu_k)$ is called the *reduced quantum trajectory*.

In the Markovian case (see [36] for a complete study), by applying a result similar to propositions 1 and 2, the evolution equation (29) is reduced to

$$\mathcal{P}\rho_{k+1} = \sum_{i=0}^{p} \frac{1}{p_i^{k+1}} \mathcal{P}\big(\overline{P}_i^{k+1} \mathcal{L}_{k+1}\big)(\mathcal{P}\rho_k)\mathbf{1}_i^{k+1}$$

and describes a Markov chain.

As we can note, equations (11), (23), (28) are very technical. In general (as in the continuous case), they are difficult to manipulate. Apart from the choice of measurement, our description is general. In the last subsection, we investigate the problems of unravelling and the question of the pure state evolution in a particular non-Markovian case.

4.3. Non pure state quantum trajectory

The success of the indirect quantum measurement theory and the quantum trajectory theory in the Markovian case lies in the fact that the indirect measurement gives rise to a pure state evolution for the reduced system. This means that if initially the state of $\mathcal{H}_0$ is pure, the reduced quantum trajectory evolves on the set of pure states of $\mathcal{H}_0$ [6, 14]. As is made precise in section 1, such a property is widely used in Monte Carlo wavefunction methods for the simulation of the quantum Markov master equation.

In this section, we start by recalling the pure state property for the discrete quantum trajectory in the Markovian case. Next, for a specific model, we investigate the conditions under which a pure state trajectory can be unravelled in the non-Markovian case.

4.3.1. Markovian unravelling. In this subsection, we show that the measurement of an observable which has different eigenvalues (with eigen-space of dimension 1) provides the unravelling result in the Markovian case.

Proposition 3. *Let (U_k) be a sequence of unitary operators describing a quantum repeated interaction setup in the Markovian case. Let A be an observable whose spectral decomposition is given by*

$$A = \sum_{i=0}^{K} \lambda_i P_i,$$

where $K + 1$ is the dimension of $\mathcal{H}$. Let $\rho_0 = |\psi_0\rangle\langle\psi_0|$ be the initial state of $\mathcal{H}_0$. Let $(\mathcal{P}\rho_k)$ be the reduced quantum trajectory describing the evolution of the state of $\mathcal{H}_0$ in the setup of quantum repeated measurements of A.

Then, there exists a random sequence of vectors (ψ_k) of $\mathcal{H}_0$ such that

$$\mathcal{P}\rho_k = |\psi_k\rangle\langle\psi_k| \otimes \bigotimes_{j=1}^{\infty} |X_0\rangle\langle X_0|. \tag{30}$$

Proof. By recursion, suppose that there exists ψ_k such that $\mathcal{P}\rho_k = |\psi_k\rangle\langle\psi_k| \otimes \bigotimes_{j=1}^{\infty} |X_0\rangle\langle X_0|$. As the evolution of $\mathcal{P}(\rho_k)$ in the Markovian case is given by

$$\mathcal{P}\rho_{k+1} = \sum_{m=0}^{K} \frac{1}{p_m^{k+1}} \mathcal{P}\big(\overline{P}_m^{k+1} \mathcal{L}_{k+1}\big)(\mathcal{P}\rho_k)\mathbf{1}_m^{k+1},$$

it is sufficient to show that each $\overline{P}_m^{k+1} \mathcal{L}_{k+1}(\mathcal{P}\rho_k)\big/ p_m^{k+1}$ can be expressed as (30). To this end, we make explicit expression of $\overline{P}_m^{k+1} \mathcal{L}_{k+1}(\mathcal{P}\rho_k)$ for all $m \in \{0, \ldots, K\}$. Let us consider a

homogeneous Markovian evolution (the non-homogeneous case can be easily adapted). With the same notation as in the proof of proposition 2, we get the following for $\mathcal{L}_{k+1}(\mathcal{P}\rho_k)$:

$$\mathcal{L}_{k+1}(\mathcal{P}\rho_k) = \sum_{ij} U_{i0}|\psi_k\rangle\langle\psi_k|U_{j0}^{\star} \otimes \bigotimes_{l=1}^{k} a_{00}^{l} \otimes a_{ij}^{k+1} \otimes \bigotimes_{l=k+2}^{\infty} a_{00}^{l}.$$

As the observable A has eigenvalues of rank 1, for all $m \in \{0, \dots, K\}$ the projector P_m is a rank-1 projector. Hence, there exists a unitary operator Q_m such that $P_m = |Q_m X_0\rangle\langle Q_m X_0|$. This allows us to write

$$\overline{P}_m^{k+1} \mathcal{L}_{k+1}(\mathcal{P}\rho_k) = \sum_{ij} U_{i0}|\psi_k\rangle\langle\psi_k|U_{j0}^{\star} \otimes \bigotimes_{l=1}^{k} a_{00}^{l}$$

$$\otimes \langle Q_m X_0, X_i\rangle\langle X_j, Q_m X_0\rangle|Q_m X_0\rangle\langle Q_m X_0| \otimes \bigotimes_{l=k+2}^{\infty} a_{00}^{l}$$

$$= \left| \sum_i \langle Q_m X_0, X_i\rangle U_{i0}\psi_k \right\rangle\!\!\left\langle \sum_j \langle Q_m X_0, X_j\rangle U_{j0}\psi_k \right|$$

$$\otimes \bigotimes_{l=1}^{k} a_{00}^{l} \otimes |Q_m X_0\rangle\langle Q_m X_0| \otimes \bigotimes_{l=k+2}^{\infty} a_{00}^{l}.$$

Hence, we get

$$\mathrm{Tr}_{T\Phi}\left[\overline{P}_m^{k+1}\mathcal{L}_{k+1}(\mathcal{P}\rho_k)\right] = \left| \sum_i \langle Q_m X_0, X_i\rangle U_{i0}\psi_k \right\rangle\!\!\left\langle \sum_j \langle Q_m X_0, X_j\rangle U_{j0}\psi_k \right| = |\theta_{k+1}\rangle\langle\theta_{k+1}|.$$

The normalization factor p_m^{k+1} gives the right expression of ψ_{k+1} and the recursion holds. $\quad\square$

4.3.2. Non Markovian unravelling.   In this section, we investigate the unravelling in the context of non-Markovian quantum repeated measurements.

    Our discussion will be based on a typical physical application, namely a two-level atom (a qubit) in contact with a chain of spins. Mathematically, for this model $\mathcal{H}_0 = \mathcal{H} = \mathbf{C}^2$. Let us make precise the description of the interaction. Let k be fixed, the interaction number k is defined by a total Hamiltonian $H_{\mathrm{tot}}(k)$ of the following form:

$$H_{\mathrm{tot}}(k) = H_0 + H_R(k) + \lambda H_I(k), \tag{31}$$

where

- The operator H_0 corresponds to the free Hamiltonian of $\mathcal{H}_0$. It is a self-adjoint operator acting non-trivially on $\mathcal{H}_0$ and acting like the identity operator on the chain.
- The operator $H_R(k)$ is the free Hamiltonian corresponding to the free evolution of the k first copies of $\mathcal{H}$. It acts non-trivially on $\bigotimes_{j=1}^{k} \mathcal{H}_j$ and like the identity operator elsewhere. If H corresponds to the free evolution of one copy of $\mathcal{H}$, we have

$$H_R(k) = \sum_{i=1}^{k} \left(I \otimes \bigotimes_{j=1}^{i-1} I \otimes H \otimes \bigotimes_{j>i+1} I \right) = \sum_{i=1}^{k} H^{(k)}. \tag{32}$$

In our context, the natural Hamiltonian of a single copy is $H = \gamma a_{00}$, where γ is a real constant.

- The self-adjoint operator $H_I(k)$ is the interaction Hamiltonian. It acts non-trivially on $\mathcal{H}_0 \otimes \bigotimes_{j=1}^{k} \mathcal{H}_j$ and like the identity operator elsewhere. The interaction Hamiltonian $H_I(k)$ is given by

$$H_I(k) = \sum_{i=1}^{k} \left(C_i(k) a_{10}^{(i)} + C_i^{\star}(k) a_{01}^{(i)} \right), \tag{33}$$

  where C_i are operators on $\mathcal{H}_0$. In other words, each copy interacts with $\mathcal{H}_0$, but they do not interact with each other. This is a simple physical case describing a non-Markovian memory effect (at each new interaction, the previous copies continue to interact).
- The term λ is a real scalar corresponding to a coupling constant.

  If the time of each interaction is τ, then the unitary operator U_k is defined as

  $$U_k = e^{i\tau H_{tot}(k)}.$$

Hence, it describes a non-Markovian quantum repeated interactions model. The non-Markovian character comes from the definition of $H_I(k)$.

Remark 1. It is worth noting that if $H_I(k)$ acts non-trivially on the tensor product of $\mathcal{H}_0$ with $\mathcal{H}_k$ and like the identity operator elsewhere, we recover a Markovian approach of quantum repeated interactions. Actually, it does not correspond directly to the previous definition of Markovian interaction but it is equivalent to it. Indeed, at the kth interaction the fact that $H_I(k)$ acts non-trivially on the tensor product of $\mathcal{H}_0$ with $\mathcal{H}_k$ and like the identity operator elsewhere, means that only $C_k(k)$ is a non-zero operator. Hence, the $(k-1)$ first copies do not interact with $\mathcal{H}_0$, while we can keep the definition of their free evolution. In the first definition of the Markovian case, we do not refer to the free evolution of the $(k-1)$ first copies which could mean that we keep a kind of memory, but by taking the partial trace this memory disappears.

The result, we shall prove is the following.

Proposition 4. *Let us consider a non-Markovian model of quantum repeated interactions defined by the interaction (31)–(33). Let A be an observable of $\mathcal{H}$.*

The unravelling property, in terms of indirect measurements of A, implies necessarily that A is of the form

$$A = \lambda_0 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \tag{34}$$

and

$$C_1(2) = C_1(2)^{\star}. \tag{35}$$

Reciprocally, let us consider the special case of an observable A of the form (34) and the special case of interaction (31)–(33) defined by

(i) $\gamma = 0$, *that is, no free evolution;*
(ii) $C_i(k) = C_i(k)^{\star}$, *for all integers i and k;*
(iii) *for all $k \geqslant 2$, $C_i(k) = 0$, for $1 \leqslant i \leqslant k-2$, that is, at the kth interaction only the copies number k and $k-1$ of $\mathcal{H}$ interact with $\mathcal{H}_0$.*

Before giving the proof, we give the interpretation of this result. As we have seen in proposition 3, under the Markovian assumption, the unravelling property holds for all observable with one rank eigenprojectors. As it is made precise in proposition 4, in the case of interaction (31)–(33), for all observables, which are not of the form (34), we do not get unravelling in terms of indirect measurements. Moreover, even if the observable is of the form

(34), the unravelling imposes a special form for the operators involved in the definition of the interaction (31)–(33). Finally, these necessary conditions allow us to express further sufficient conditions in order to obtain a situation in which we find the unravelling property.

The crucial fact, which prevents the unravelling in the non-Markovian case, is the fact that at the kth step the measurement involves only the copy number k of $\mathcal{H}$ whereas the evolution involves all the previous copies. The interaction with the other copies of $\mathcal{H}$, which are not measured, produces entanglement and prevents the description of the evolution of the small system in terms of pure states.

Proof. Our strategy to study the unravelling question in this context is the following one. We start by considering the two first interactions, that is $\mathcal{H}_0$ coupled with $\mathcal{H}_1$ and next with $\mathcal{H}_1 \otimes \mathcal{H}_2$. With a general approach, we show that the condition of unravelling imposes assumptions on the definition of the interaction. Next, we translate these assumptions in terms of the interaction model (31). Finally, by considering the small interaction time τ, we show that unravelling imposes to 'forget' the interaction with $\mathcal{H}_1$ during the second interaction, except for a very special case related with conditions (34) and (35). Let us stress that forgetting the interaction with $\mathcal{H}_1$ during the second interaction corresponds to the Markovian assumption. Next, we investigate the special case of the proposition.

The space describing the two first interactions is $\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$. Let

$$U_1 = \sum_{i,j=0,1} U_{i,j} a_{ij}^1 \otimes I$$

be a unitary operator describing the first interaction and let

$$U_2 = \sum_{i,j=0,1} \sum_{k,l=0,1} U_{i,j,k,l} a_{ij}^1 \otimes a_{kl}^2$$

be a unitary describing the second interaction. Let $A = \lambda_0 P_0 + \lambda_1 P_1$ be the measured observable and let $\rho = |\psi\rangle\langle\psi|$ be the initial state of $\mathcal{H}_0$.

Concerning the observable A, there exist unitary matrices $Q_m, m = 0, 1$ such that $P_m = |Q_m X_0\rangle\langle Q_m X_0|$. According to the unravelling result in the Markovian case, the first random state ρ_1 on $\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$ can take one of the values

$$\frac{1}{Z_1(m)} |\psi_1(m)\rangle\langle\psi_1(m)| \otimes |Q_m X_0\rangle\langle Q_m X_0| \otimes |X_0\rangle\langle X_0|, \qquad m = 0, 1,$$

where $\psi_1(m) = \left| \sum_i \langle Q_m X_0, X_i\rangle U_{i0}\psi \right\rangle$ and Z_1 corresponds to the normalization factor. Hence the state $\mathcal{P}\rho_1$ is a pure state. For the second interaction and the second measurement, we introduce the following notation:

$$\mathcal{L}_{u0} = \left(\sum_{i,j=0,1} U_{iju0} a_{ij}^1 \right) \otimes a_{u0}^2, u = 0, 1.$$

In the same way, the second state ρ_2 can take one of the values

$$\rho_2(m, n) = \frac{1}{Z_2(m, n)} \left(\sum_{u,v} \langle Q_n X_0, X_u\rangle\langle X_v, Q_n X_0\rangle \mathcal{L}_{u0} \left(\psi_1(m) \otimes |Q_m X_0\rangle\langle Q_m X_0|\right) \mathcal{L}_{v0}^\star \right)$$
$$\otimes |Q_n X_0\rangle\langle Q_n X_0|,$$

with $(m, n) \in \{0, 1\}^2$. The accurate computation of $\mathcal{P}\rho_2(m, n)$ gives

$$\mathcal{P}\rho_2(m, n) = \frac{1}{Z_2(m, n)} (|\mathcal{H}_0(m, n)\psi_1(m)\rangle\langle\mathcal{H}_0(m, n)\psi_1(m)|$$
$$+ |\mathcal{H}_1(m, n)\psi_1(m)\rangle\langle\mathcal{H}_1(m, n)\psi_1(m)|) \otimes a_{00}^1 \otimes a_{00}^2,$$

where

$$\mathcal{H}_i(m,n) = \sum_{u,v} \langle Q_n X_0, X_u \rangle \langle X_v, Q_m X_0 \rangle U_{i,u,v,0}, \quad i = 0, 1.$$

As a consequence, unravelling is possible if the following condition is satisfied:

$$|\mathcal{H}_0(m,n)\psi_1(m)\rangle\langle\mathcal{H}_0(m,n)\psi_1(m)| + |\mathcal{H}_1(m,n)\psi_1(m)\rangle\langle\mathcal{H}_1(m,n)\psi_1(m)|$$
$$= |\psi_2(m,n)\rangle\langle\psi_2(m,n)|,$$

for some vector $\psi_2(m,n)$. If we want a rank-1 projector to be expressed as the sum of two rank-1 projectors, this imposes that there exist constants μ and ν such that

$$\nu\mathcal{H}_0(m,n)\psi_1 + \mu\mathcal{H}_1(m,n)\psi_1 = 0.$$

In general, μ and ν will depend on n, m and also on ψ_1. Here, we assume that there exist non-zero constants $\mu(m,n)$ and $\nu(m,n)$ such that

$$\nu(m,n)\mathcal{H}_0(m,n) + \mu(m,n)\mathcal{H}_1(m,n) = 0. \tag{36}$$

This assumption will be justified after the proof.

Now, our aim is to show how the condition (36) is connected to the possibility of obtaining unravelling in the case of the interaction (31). The condition (36) imposes a relation between the coefficients $U_{i,j,l}$ of the unitary operator U_2. Let us express these coefficients in the interaction model (31). The total Hamiltonian for two interactions is defined by

$$H_{tot} = H_0 \otimes I \otimes I + I \otimes \gamma a_{00} \otimes I + I \otimes I \otimes \gamma a_{00} + \lambda(C_1(2) \otimes a_{10} \otimes I + C_1^\star(2) \otimes a_{01} \otimes I$$
$$+ C_2(2) \otimes I \otimes a_{10} + C_2^\star(2) \otimes I \otimes a_{01}).$$

In order to describe the coefficients of U_2 in terms of H_{tot}, we introduce an appropriate basis of $\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$, which is $X_0 \otimes X_0 \otimes X_0, X_1 \otimes X_0 \otimes X_0, X_0 \otimes X_1 \otimes X_0, X_1 \otimes X_1 \otimes X_0, X_0 \otimes X_0 \otimes X_1, X_1 \otimes X_0 \otimes X_1, X_0 \otimes X_1 \otimes X_1, X_1 \otimes X_1 \otimes X_1$. Hence, we can write

$$H_{\text{tot}} = \begin{pmatrix} H_0 + 2\gamma I & \lambda C_1(2) & \lambda C_2(2) & 0 \\ \lambda C_1^\star(2) & H_0 + \gamma I & 0 & \lambda C_2(2) \\ \lambda C_2^\star(2) & 0 & H_0 + \gamma I & \lambda C_1(2) \\ 0 & \lambda C_2^\star(2) & \lambda C_1^\star(2) & H_0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} U_{0,0,0,0} & U_{0,1,0,0} & * & * \\ U_{1,0,0,0} & U_{1,1,0,0} & * & * \\ U_{0,0,1,0} & U_{0,1,1,0} & * & * \\ U_{1,0,1,0} & U_{1,1,1,0} & * & * \end{pmatrix}.$$

We use the symbol $*$ to complete the coefficients of U_2 because these coefficients do not appear in the final result. Now, by studying the asymptotic expansion of $U_2 = e^{i\tau H_{\text{tot}}}$ in term of τ, one can find operators $L_{i,j,k,l}(\tau)$ such that

$$U_{0,0,0,0} = I + \tau L_{000}(\tau), \qquad U_{1,1,0,0} = I + \tau L_{1100}(\tau), \tag{37}$$

$$U_{0,1,0,0} = \tau C_1(2) L_{0,1,0,0}(\tau), \qquad U_{1,0,0,0} = \tau C_1^\star(2) L_{1,0,0,0}(\tau), \tag{38}$$

$$U_{0,0,1,0} = \tau C_2^\star(2) L_{1,0,0,0}(\tau), \qquad U_{1,1,1,0} = \tau C_2(2) L_{1,0,0,0}(\tau), \tag{39}$$

$$U_{0,1,0,1} = \tau^2 C_1(2) L_{1,0,0,0}(\tau), \qquad U_{1,0,1,0} = \tau^2 C_1^\star(2) L_{1,0,0,0}(\tau). \tag{40}$$

Furthermore, each operator $L_{i,j,k,l}(\tau)$ converges to a non-zero operator when τ goes to zero.

Now, by comparing this asymptotic description in terms of τ and the condition (36), we investigate the question of unravelling. The result depends on the form of the observable A. There are two different situations.

The first situation corresponds to the case where $P_0 = |X_0\rangle\langle X_0|$ (this is equivalent to the case where $P_1 = |X_0\rangle\langle X_0|$). In this case we have the corresponding $Q_0 = I$, then $\langle X_0, Q_0 X_o \rangle = 1$ and the condition (36) for $m = n = 0$ gives

$$\nu(0,0)U_{0,0,0,0} = \mu(0,0)U_{1,0,0,0}.$$

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Hence, the asymptotic conditions (37)–(40) impose $C_1(2) = 0$. This situation corresponds to the Markovian case.

As a consequence, for any observable where $P_0 = |X_0\rangle\langle X_0|$ (or P_1), the unravelling condition imposes the Markovian approach since the memory of the interaction with the first copy disappears. Indeed, by recursion, for the third interaction and measurement, a similar reasoning shows that the unravelling condition imposes to forget the memory of the second copy and so on.

The second situation concerns the case where $0 < \langle X_0, Q_m X_0 \rangle < 1, m = 0, 1$. Indeed, if for $m \in \{0, 1\}$, we have $\langle X_0, Q_m X_0 \rangle \in \{0, 1\}$, which corresponds to the first case. Now, the condition (36) and the asymptotic conditions (37)–(40) give

$$(\nu(m, n)\langle Q_n X_0, X_0 \rangle\langle X_0, Q_m X_0 \rangle - \mu(m, n)\langle Q_n X_0, X_0 \rangle\langle X_1, Q_m X_0 \rangle)I = \tau W(\tau),$$

where $W(\tau)$ converges to an operator when τ goes to zero. Hence, it imposes that

$$\frac{\nu(m, n)}{\mu(m, n)} = \frac{\langle X_1, Q_m X_0 \rangle}{\langle X_0, Q_m X_0 \rangle}. \tag{41}$$

Now, let us deal with the terms in τ^2. If $C_1(2) \neq C_1(2)^\star$, that is if the condition (35) is not satisfied, we obtain an equality of the following form:

$$\tau^2 Z(\tau) = \tau Y(\tau),$$

where $Z(\tau)$ and $Y(\tau)$ converge to non-zero operators, when τ goes to zero. As a consequence, if $C_1(2) \neq C_1(2)^\star$, we cannot get unravelling.

Let us now suppose that $C_1(2) = C_1(2)^\star$. The condition (36) and the asymptotic conditions (37)–(40) give

$$\tau^2(\nu(m, n)\langle Q_n X_0, X_1 \rangle\langle X_1, Q_m X_0 \rangle - \mu(m, n)\langle Q_n X_0, X_1 \rangle\langle X_0, Q_m X_0 \rangle)C_1(2)N(\tau)$$
$$= \tau M(\tau),$$

where $N(\tau)$ and $M(\tau)$ converge to non-zero operators when τ goes to zero. It imposes the condition

$$\frac{\nu(m, n)}{\mu(m, n)} = \frac{\langle X_0, Q_m X_0 \rangle}{\langle X_1, Q_m X_0 \rangle}, \qquad \text{or} \qquad C_1(2) = 0. \tag{42}$$

The case $C_1(2) = 0$ corresponds to the Markovian approach and the result in section 4.3.1 imposes naturally the unravelling result. Otherwise, the condition (41) combined with the condition (42) implies that

$$\langle X_0, Q_m X_0 \rangle^2 = \langle X_1, Q_m X_0 \rangle^2, \qquad m = 0, 1.$$

It is easy to note that this condition corresponds to special observables of the form (34). This means that in the non-Markovian case with the interaction model (31), the unravelling result cannot be obtained if the measured observable is not of the form (34) with $C_1(2) = C_1(2)^\star$.

Reciprocally, we can now exhibit a special model of interaction (31) that allows unravelling. We consider that for all $(i, k) \in \mathbf{N}$, we have $C_i(k) = C_i(k)^\star$ (this corresponds to the assumption $C_1(2) = C_1(2)^\star$). Furthermore, we assume that there is not free evolution concerning the infinite chain, that is $\gamma = 0$. Moreover, to use directly the previous discussion, we suppose that at each interaction only two copies of $\mathcal{H}$ interact with $\mathcal{H}_0$. Mathematically, for $k \geqslant 2$, at the kth interaction, we consider that only the copies number k and number $k - 1$ interact with $\mathcal{H}_0$. In other words, for $k \geqslant 2$, at the kth interaction, we have $C_i(k) = 0$ for all $1 \leqslant i \leqslant k - 2$. Now, for an observable of the form (34), by computing $e^{i\tau H_{\text{tot}}}$, it is easy to see that

$$\mathcal{H}_0(m, n) = \mathcal{H}_1(m, n), \qquad m, n = 0, 1$$

and the condition (36) is fulfilled. As only the last incoming copy and the previous one interact with $\mathcal{H}_0$ at each interaction, a recursive reasoning shows that we obtain unravelling in this case.   □

Let us finish by justifying that the condition

$$\nu \mathcal{H}_0(m, n)\psi_1 + \mu \mathcal{H}_1(m, n)\psi_1 = 0,$$

for constant ν and μ depending on n, m and ψ_1 implies that

$$\nu(m, n)\mathcal{H}_0(m, n) + \mu(m, n)\mathcal{H}_1(m, n) = 0,$$

without dependence in ψ_1. This result stems from the following proposition in linear algebra.

Proposition 5. *Let $D_1 = \{\psi \in \mathbf{C}^2 / \|\psi\| = 1\}$ be the unit disk. Let ϕ be a linear application such that for all $|x\rangle \in D_1$ there exist λ such that $\phi(|x\rangle) = \lambda|x\rangle$, then there exists λ such that for all $|x\rangle$, we have $\phi(|x\rangle) = \lambda|x\rangle$, that is $\phi = \lambda I$.*

In our case, we aim to apply this result to $\mathcal{H}_0(m, n)^{-1}\mathcal{H}_1(m, n)$ for appropriate m and n. Indeed, it is worth noting that for τ small enough, there always exists $(m, n) \in \{0, 1\}$ such that the operator $\mathcal{H}_0(m, n)$ is invertible. Indeed there always exists $(m, n) \in \{0, 1\}$ such that $\mathcal{H}(m, n)$ is of the form $\alpha I + \tau F(\tau)$, where $F(\tau)$ converges when τ goes to zero and α is non-null.

Remark 2. For all non-diagonal observables this property is satisfied for all couples (m, n). For the diagonal observable, we can consider the case $m = n = 0$.

To conclude, it remains to prove that random vectors ψ_1 cover all the unit disk. This is justified as follows. The rules of the first interaction and the first measurement give rise to a random transformation $\mathbf{\Lambda}$ from D_1 to D_1, which maps ψ_0 in a random unit vector ψ_1.

For all observable A, it is worth noting that for a τ small enough there exists $m \in \{0, 1\}$ such that the operator $\sum_i \langle Q_m X_0, X_i \rangle U_{i0}$ is invertible (for the non-diagonal observable for all m this property is satisfied, for the diagonal observable it corresponds to the case $m = 0$). As we must consider all the possible results for the measurement, in any case, there exists at least one possibility that the range of $\mathbf{\Lambda}$ is D_1 and the result holds.

5. Summary and conclusion

We have presented a quantum repeated interactions model 'with memory' and we have discussed the general situation of a non-Markovian interaction between a small system and an infinite chain. We have shown that this approach is tailored to give a clear description of indirect measurement. Also, the usual projection operation techniques were used to derive the equations of evolution of the reduced system. Without and with measurement, these equations reproduce discrete versions of the continuous time evolution equations which are already considered in the literature.

The question of unravelling of non-Markovian master equations in terms of indirect measurements has been highly debated in the literature. In order to rigorously address this interesting problem we have applied our discrete repeated interaction approach to a simple, but insightful, model. We were able to show, that in general imposing an unravelling condition is equivalent to imposing the Markov condition, thus making non-Markovian unravellings non-relevant. Nevertheless, for very special observables and interaction models non-Markovian indirect measurement unravellings were found to be possible.

Let us stress that we concentrated on the scheme where, at each interaction, only the new incoming copy of $\mathcal{H}$ is observed. From the physical point of view, when for example a new photon interacts with the small system, a measurement is performed only on this new incoming photon. This way, only one measurement apparatus is involved. One can also consider a procedure, where at each interaction, all the interacting copies are observed simultaneously. Such a context could be linked with the notion of *retarded* observable [17] and one could expect more general results on unravelling in terms of indirect measurement. This will be discussed in future works.

As a conclusion, our main objective was to demonstrate that the discrete repeated interaction approach is an interesting framework to address fundamental questions regarding non-Markovian evolution of quantum systems. In particular, we have shown that pertinent interpretations of non-Markovian unravellings can be found.

References

[1] Alicki R and Fannes M 2001 *Quantum Dynamical Systems* (Oxford: Oxford University Press)
[2] Attal S 2009 *Quant. Noises* in preparation
[3] Attal S and Pautrat Y 2006 From repeated to continuous quantum interactions *Ann. Inst. Henri Poincaré* **7** 59–104
[4] Attal S and Pautrat Y 2005 From $(n + 1)$-level atom chains to n-dimensional noises *Ann. Inst. Henri Poincaré Probab. Stat.* **41** 391–07
[5] Barchielli A and Holevo A S 1995 Constructing quantum measurement processes via classical stochastic calculus *Stoch. Process. Appl.* **58** 293–317
[6] Barchielli A and Lupieri G 2004 Instrumental processes, entropies, information in quantum continual measurements *Quantum Inf. Comput.* **4** 437
[7] Barchielli A and Gregoratti M 2009 *Quantum Trajectories and Measurements in Continuous Time* in preparation
[8] Bouten L, van Handel R and James M 2007 An introduction to quantum filtering *SIAM J. Control Opt.* **46** 2199–41
[9] Bouten L, Guta M and Maassen H 2004 Stochastic Schrödinger equations *J. Phys. A: Math. Gen.* **37** 3189–09
[10] Breuer H-P 2007 Non-Markovian generalization of the Lindblad theory of open quantum systems *Phys. Rev. A* **75** 022103
[11] Breuer H-P and Vacchini B 2008 Quantum semi-Markov processes *Phys. Rev. Lett.* **101** 140402
[12] Breuer H-P, Kappler B and Petruccione F 2001 The time-convolutionless projection operator technique in the quantum theory of dissipation and decoherence *Ann. Phys.* **291** 36–70
[13] Breuer H-P, Kappler B and Petruccione 1999 Stochastic wave function method for non-markovian quantum master equations *Phys. Rev. A* **59** 16–33
[14] Breuer H-P and Petruccione F 2002 *The Theory of Open Quantum Systems* (New York: Oxford University Press)
[15] Breuer H-P and Petruccione F 1997 Dissipative quantum systems in strong laser: stochastic wave-function method and Floquet theory *Phys. Rev. A* **55** 3101–16
[16] Davies E B 1976 *Quantum Theory of Open Systems* (London: Academic)
[17] Diósi L 2008 Non-Markovian continuous quantum measurement of retarded observables *Phys. Rev. Lett.* **100** 080401
[18] Diósi L, Gisin N and Strunz W T 1999 Open system dynamics with non-Markovian quantum trajectories *Phys. Rev. Lett.* **82** 1801–05
[19] Diósi L, Gisin N and Strunz W T 1998 Non-Markovian quantum state diffusion *Phys. Rev. A* **58** 1699–12
[20] Diósi L and Strunz W T 1997 The non-Markovian stochastic Schrödinger equation for open systems *Phys. Lett. A* **235** 569–73
[21] Gambetta J, Askerud T and Wiseman H M 2004 Jumplike unravellings for non-Markovian open quantum systems *Phys. Rev. A* **69**
[22] Gambetta J and Wiseman H M 2003 Interpretation of non-Markovian stochastic Schrödinger equation as a hidden-variable theory *Phys. Rev. A* **68** 062104
[23] Gambetta J and Wiseman H M 2002 A perturbative approach to non-Markovian stochastic Schrödinger equations *Phys. Rev. A* **66** 052105

[24] Gambetta J and Wiseman H M 2002 Non-Markovian stochastic Schrödinger equations: generalization to real-valued noise using quantum measurement theory *Phys. Rev.* A **66** 012108

[25] Gardiner C W and Zoller P 2004 *Quantum Noise. A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics* (Springer Series in Synergetics) 3rd edn (Berlin: Springer)

[26] Gleyzes S, Kuhr S, Guerlin C, Bernu J, Delglise S, Hoff U B, Brune M, Raimond J-M and Haroche S 2007 Quantum jumps of light recording the birth and death of a photon in a cavity *Nature* **446** 297–300

[27] Gorini V, Kossakowski A and Sudarshan E C G 1976 Completely positive dynamical semigroups of N-level systems *J. Math. Phys.* **17** 821–5

[28] Holevo A S 1988 *A Noncommutative Generalization of Conditionally Positive Definite Functions* (Berlin: Springer) pp 128–48

[29] Lindblad G 1979 Non-Markovian quantum stochastic processes and their entropy *Commun. Math. Phys.* **65** 281–94

[30] Lindblad G 1976 On the generators of quantum dynamical semigroups *Commun. Math. Phys.* **48** 119–30

[31] Luczka J 2005 Non-Markovian stochastic processes: colored noise *Chaos* **15** 026107

[32] Mølmer K, Castin Y and Dalibard J 1993 Monte Carlo wavefunction method in quantum optics *J. Opt. Soc. Am.* B **10** 524–38

[33] Parthasarathy K R 1992 *An Introduction to Quantum Stochastic Calculus* (*Monographs in Mathematics* vol 85) 85 (Basel: Birkhuser Verlag)

[34] Pellegrini C 2008 Existence, uniqueness and approximation of stochastic Schrödinger equation: the diffusive case *Ann. Probab.* **36** 2332–53

[35] Pellegrini C 2008 Existence, uniqueness and approximation for stochastic Schrödinger equation: the Poisson case arXiv:0709.3713

[36] Pellegrini C 2008 Markov chains approximations of jump-diffusion quantum trajectories arXiv:0803.2593

[37] Maniscalco S and Petruccione F 2006 Non-Markovian dynamics of a qubit *Phys. Rev.* A **73** 012111

[38] Wiseman H M and Gambetta J 2008 Pure-state quantum trajectories for general non-Markovian systems do not exist *Phys. Rev. Lett.* **101** 140401

[39] Sudarshan E C G 2003 Evolution and decoherence in finite level systems *Chaos Solitons Fractals* **16** 369–79

[40] Zurek W H 1991 Decoherence and the transition from quantum to classical *Phys. Today* **44** 36–44

[41] Zwanzig R 1982 Non-Markovian diffusion in a one-dimensional disordered lattice *J. Stat. Phys.* **28** 127–33